Sensor and Simulation Notes

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Simple Mathematical Models of Reflector IRAs with 2-Arm and 4-Arm Feeds

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Abstract

While simple mathematical models are available for reflector Impulse Radiating Antennas (IRAs), they have not yet been cast into the form of the recently derived antenna equation. Doing so proves instructive, as it reveals properties that were previously obscured. We provide here simple mathematical models of 2-arm and 4-arm reflector IRAs, with feed arms positioned either vertically, or at ±45° to vertical. We cast these models into the form of the antenna equation, and we provide antenna transfer function and realized gain for each antenna. Somewhat to our surprise, our mid-band estimates for the two configurations yield the same antenna transfer function and realized gain on boresight. We provide an explanation for why that is a reasonable result. We also find that both 2-arm and 4-arm designs have an aperture efficiency of 30% for the typical values of input impedance of 400 Ω and 200 Ω, respectively. In an appendix, we add the prepulse to these simple models.

I. Introduction

We provide here a simple estimate of fields and gain for a 2-arm IRA and a 4-arm IRA. In the case of a 4-arm feed, the feed arms are positioned orthogonal to each other at ±45° to vertical. The input impedance of the 2-arm IRA is nominally 400 Ω, and that of the 4-arm IRA is 200 Ω. Figure 1 shows the two configurations. We use these models to provide antenna impulse response, antenna transfer function, and realized gain.

In practice, one generally uses a 4-arm configuration. However, it is useful to calculate both the 2-arm and 4-arm cases, as they provide a consistency check on the validity of both calculations.

In this paper, we use the following symbols,

$E_{rad,2}(t)$	far electric field radiated on boresight for a 2-arm IRA
$E_{rad,4}(t)$	far electric field radiated on boresight for a 4-arm IRA
$V(t)$	total voltage across the feed arms
D	diameter of the reflector
c	speed of light in free space, ≈ 0.3 m/ns
r	distance from antenna to observation point on boresight
η	impedance of free space, ≈ 377 Ω.
Z_{o2}	impedance of a 2-arm feed, typically 400 Ω.
Z_{o4}	impedance of a 4-arm feed, typically 200 Ω.
f_{g2}	Z_{o2}/η
f_{g4}	Z_{o4}/η
f	frequency
s	Laplace transform variable, $s = j\omega = j\,2\pi f$
$h(t)$	antenna impulse response, time domain
$\tilde{h}$, $h(s)$	antenna transfer function, in the frequency or Laplace domain
$\delta_a(t)$	approximate Dirac delta function, as defined in [3] and [6]
$u(t)$	Heaviside step function
G_r	realized gain

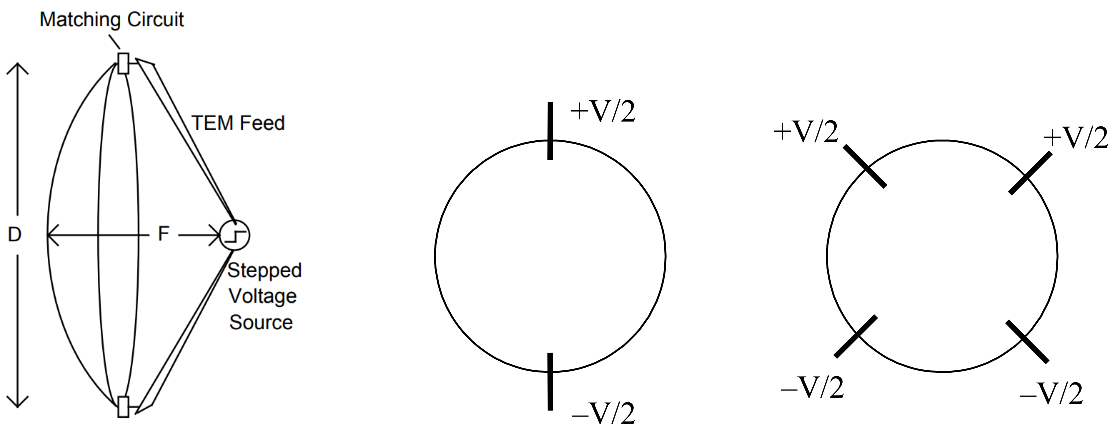

Figure 1. Sketch of a 2-arm IRA (left and center) and a 4-arm IRA (right).

II. 2-Arm Configuration

We wish to calculate the boresight fields, antenna transfer function, and gain of two configurations of IRAs. We begin with the 2-arm configuration. From [1], [2], [3], we have the electric field radiated on boresight in the far field as

$$E_{rad,2}(t) \approx \frac{D}{4\pi r c f_{g2}} \frac{dV(t)}{dt} \tag{1}$$

We ignore for the moment the prepulse since it has little effect on the peak field. We treat the prepulse in Appendix A. The subscript 2 indicates that this is for the 2-arm case.

To find the antenna transfer function, we cast this into the form of the antenna equation [4].

$$\frac{r E_{rad,2}(t)}{\sqrt{\eta}} \approx \frac{D}{4\pi c\sqrt{f_{g2}}} \frac{d}{dt}\left(\frac{V(t)}{\sqrt{Z_{o2}}}\right) , \tag{2}$$

or in the frequency (Laplace) domain

$$\frac{r \tilde{E}_{rad,2}}{\sqrt{\eta}} \approx \frac{s D}{4\pi c\sqrt{f_{g2}}} \frac{\tilde{V}}{\sqrt{Z_{o2}}} , \tag{3}$$

where the tilde indicates a frequency domain quantity in the Laplace domain, and s is the Laplace transform variable.

Consider now the antenna equation [4, eqn. (3)].

$$\begin{bmatrix} \tilde{b} \\ \tilde{\xi} \end{bmatrix} = \begin{bmatrix} \tilde{\Gamma} & \tilde{h} \\ s\,\tilde{h}/(2\pi c) & \tilde{\ell} \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{\zeta} \end{bmatrix} . \tag{4}$$

The equation of interest is the transmission equation, which is associated with the lower left element of the matrix. Thus,

$$\tilde{\xi} = \frac{s\,\tilde{h}}{2\pi c}\,\tilde{a} \quad , \tag{5}$$

where

$$\tilde{\xi} = \frac{r\,\tilde{E}_{rad,2}}{\sqrt{\eta}} \qquad , \qquad \tilde{a} = \frac{\tilde{V}}{\sqrt{Z_{o2}}} \quad . \tag{6}$$

Comparing (3) and (5), we find the antenna transfer function for the 2-arm IRA,

$$\tilde{h}_2 \approx \frac{D}{2\sqrt{f_{g2}}} \quad . \tag{7}$$

Note that the antenna transfer function, $\tilde{h}_2$, is simply a constant in the frequency domain. This is what we expect for a focused aperture antenna. Note further that this is valid only at mid-band, as we have ignored the prepulse at low frequencies, and imperfections in the feed point at high frequencies. Note also that $h_2(s)$ may also be used to estimate the received voltage due to an incident field, using the antenna equation (4).

Now realized gain is calculated from the antenna transfer function [4, eqn.(5)]

$$G_{r2}(s) = \frac{4\pi}{\lambda^2}\left|\tilde{h}_2\right|^2 \quad . \tag{8}$$

Combining (7) and (8), we have

$$G_{r2}(s) = \frac{\pi}{f_{g2}}\frac{D^2}{\lambda^2} = \frac{\pi D^2}{f_{g2}\,c^2} f^2 \quad . \tag{9}$$

In the second expression above, we have just used $\lambda = c\,/\,f$. Again, this is valid only at mid-band. For the typical value of $f_{g2} = 400\ \Omega\,/\,\eta$, and $\eta = 120\ \pi\ \Omega$, we have

$$G_{r2} \;=\; 0.3\,\pi^2 \frac{D^2}{\lambda^2} \quad . \tag{10}$$

On the other hand, an ideal circular aperture with area A has a realized gain of

$$G_{ri} \;=\; 4\pi \frac{A}{\lambda^2} \;=\; \pi^2 \frac{D^2}{\lambda^2} \quad , \tag{11}$$

where we have used $A = \pi D^2/4$. This represents an ideal that is not achieved in practice, but it provides a useful point of comparison. Comparing the above two equations,

$$G_{r2} \;=\; 0.3\,G_{ri} \quad . \tag{12}$$

So, an ideal 2-arm IRA with feed impedance of 400 Ω has 0.3 times the gain of an ideal circular aperture, or a 30% aperture efficiency. Alternatively, the effective aperture is 0.3 A. This would be low for a narrowband antenna, but that is the price one sometimes pays to achieve a broadband design.

III. 4-Arm Configuration

If we have a 4-arm configuration, we add a second pair of arms in the plane of symmetry of the first, so it only minimally affects the fields of the original 2-arm configuration. The radiated fields from the two pairs of feed arms add in quadrature in the far field, so the electric field is increased by a factor of $\sqrt{2}$ over that shown in (1) for the single pair of arms. The radiated field then becomes

$$E_{rad,4}(t) \;\approx\; \frac{\sqrt{2}\,D}{4\pi\, r\, c\, f_{g2}} \frac{dV(t)}{dt} \;\approx\; \frac{D}{4\sqrt{2}\,\pi\, r\, c\, f_{g4}} \frac{dV(t)}{dt} \quad . \tag{13}$$

Here we used $f_{g2} = 2\,f_{g4}$.

Once again, we cast this into a format compatible with the antenna equation,

$$\frac{r\,E_{rad,4}(t)}{\sqrt{\eta}} \;\approx\; \frac{D}{4\sqrt{2 f_{g4}}\,\pi\, c} \frac{d}{dt}\left(\frac{V(t)}{\sqrt{Z_{o4}}} \right) \quad . \tag{14}$$

or in the frequency domain as

$$\frac{r\,\tilde{E}_{rad,4}}{\sqrt{\eta}} \approx \frac{s\,D}{4\sqrt{2\,f_{g4}}\,\pi\,c}\,\frac{\tilde{V}}{\sqrt{Z_{o4}}} \quad . \tag{15}$$

Comparing (15) and (5), we find for the 4-arm IRA,

$$\tilde{h}_4 \approx \frac{D}{2\sqrt{2 f_{g4}}} \quad . \tag{16}$$

This is a surprisingly interesting result. Using $f_{g4} = ½ f_{g2}$, and comparing to (7), we find

$$\tilde{h}_4 = \tilde{h}_2 \ . \tag{17}$$

Thus, the antenna transfer function is the same for the 4-arm case as it is for the 2-arm case. We provide some discussion in the next section about why this is the expected result.

Because of (17), the gains for the two cases are also equal. Thus

$$G_{r4}(s) = \frac{\pi}{2 f_{g4}}\frac{D^2}{\lambda^2} = \frac{\pi\,D^2}{2 f_{g4}\,c^2}\,f^2 \quad . \tag{18}$$

As before, this is an approximation valid at only at mid-band. At high frequency, the details of the feed point will force the gain to roll off. At low frequency, the effect of the prepulse will become apparent. Since this gain is the same as that for the 2-arm configuration, the aperture efficiency is also the same, 30%.

IV. Discussion

At first, it seemed surprising that the 2-arm configuration provides roughly the same mid-band performance on boresight as the 4-arm configuration. Two questions immediately arise. 1) Is that a reasonable result? and 2) Why would one bother adding the second pair of arms if one gets the same result?

We start with the first question, the reasonableness of the result. We note that in the 4-arm case, the fields generated by each pair of arms are orthogonal, so the feed arms accept twice the power as in the 2-arm case. Furthermore, the radiated electric field is increased by $\sqrt{2}$ because the fields from each pair of orthogonal feed arms add in quadrature. So, the radiation intensity increases by a factor of 2. Realized gain is proportional to the ratio of the radiation intensity to the power accepted by the antenna, so the two effects cancel out. This results in the same realized gain for both cases. If realized gain remains the same, then the antenna transfer function, $h(s)$, must also remain the same. So, the performance of the two configurations is approximately the same on boresight at mid-band.

Next, we address the second question, the usefulness of the second pair of feed arms. We note three items. First, the impedance of sources tends to be lower than the impedance of antennas. Anything one can do to lower the antenna input impedance makes it easier to match the antenna impedance to the source impedance. Second, the radiated fields for a 2-arm configuration have a fan-shaped beam, wider in the H-plane and narrower in the E-plane [5]. The beam shape for a 4-arm configuration is much closer to equal in the E- and H-planes. One normally prefers to have equal beamwidths in the E- and H-planes. Third, a common balun configuration is available at low power to match a 50-ohm feed to a 200-ohm antenna; no such balun is available for a 400-ohm antenna. For the above three reasons, 200-ohm antennas with four arms will continue to be popular.

Finally, we note that many people are also interested in the prepulse. For completeness, we add that to our mathematical model in Appendix I.

V. Conclusion

We provided here simple calculations showing that the boresight performance of classical 2-arm and 4-arm reflector IRAs is approximately the same at mid-band and on boresight. An explanation for why that is reasonable is also provided. Both designs have an aperture efficiency of 30% for the typical values of input impedance of 400 Ω and 200 Ω, respectively. A more complete model, including the prepulse, is in Appendix A.

References

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Appendix A. Extending the Mathematical Model to Include the Prepulse

Until now, we have used expressions as simple as possible. However, for completeness, we provide the full version of the IRA mathematical model on boresight in the far field, including the prepulse. From [1], [2], [3], and [4], we have for the 2-arm configuration, in both the time and frequency domains,

$$\begin{aligned} h_2(t) &= \frac{D}{2\sqrt{f_{g2}}}\left\{\delta_a(t-t_o) - \frac{c}{2F}\left[u(t)-u(t-t_o)\right]\right\} \\ \tilde{h}_2 &= \frac{D}{2\sqrt{f_{g2}}}\left\{e^{-st_o} - \frac{c}{2F}\frac{1}{s}\left[1-e^{-st_o}\right]\right\} \\ t_o &= 2F/c \end{aligned} \qquad . \qquad \text{(A-1)}$$

Here, $\delta_a(t)$ is the approximate Dirac delta function as defined in [3] and [6]. This is a function whose area is unity, but which has a finite magnitude. In the limit as $r \to \infty$ it becomes a true Dirac delta function. Furthermore, $u(t)$ is the Heaviside step function. Finally, $t_o = 2F/c$ is the round-trip transit time between the parabolic focus and the center of the paraboloidal reflector.

A sketch of this impulse response in the time domain is shown in Figure A-1. The above equations assume that the prepulse starts at $t = 0$. A notable feature of this impulse response is that the area of the impulse is the same as the area of the prepulse, which was shown in [7] to be a very good approximation.

It may be of interest to clarify the regions of validity of the above model. The impulsive portion, proportional to $\delta_a(t)$, is valid only in the far field. On the other hand, the prepulse, consisting of a difference of step functions, is valid in both the near and far fields. This may be important if one is characterizing and tuning a transmitter in a laboratory setting, where far-field measurement is impractical.

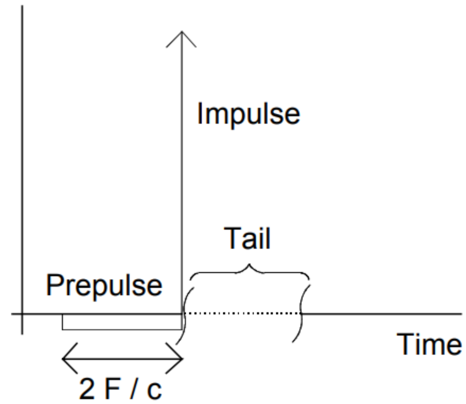

Figure A-1. Sketch of the impulse response of an IRA.

As we have shown already, the result for the 4-arm IRA is the same as that for the 2-arm IRA, so there is no need to recalculate it. However, one must use the f_g value for the 2-arm IRA (typically ~400 Ω) to get the right answer.

If we combine the above with the antenna equation (4), the radiated fields in the time and frequency domains become

$$
\begin{aligned}
\frac{r\,E_{rad,2}(t)}{\sqrt{\eta}} &= \frac{D}{4\pi\, c\sqrt{f_{g2}}}\left\{\frac{d}{dt}\left(\frac{V_{inc}(t-t_o)}{\sqrt{Z_{o2}}}\right) - \frac{c}{2F}\left[\frac{V_{inc}(t)}{\sqrt{Z_{o2}}} - \frac{V_{inc}(t-t_o)}{\sqrt{Z_{o2}}}\right]\right\} \\
\frac{r\,\tilde{E}_{rad,2}}{\sqrt{\eta}} &= \frac{D}{4\pi\, c\sqrt{f_{g2}}}\left\{s\left(\frac{\tilde{V}_{inc}e^{-st_o}}{\sqrt{Z_{o2}}}\right) - \frac{c}{2F}\frac{\tilde{V}_{inc}}{\sqrt{Z_{o2}}}\left[1-e^{-st_o}\right]\right\}
\end{aligned}
\quad . \qquad \text{(A-2)}
$$

In the above, the lossless time delay factor, $e^{-sr/c}$, is omitted for simplicity. In reception, the equations are

$$
\begin{aligned}
\frac{V_{rec,2}(t)}{\sqrt{Z_c}} &= \frac{D}{2\sqrt{f_{g2}}}\left\{\frac{E_{inc}(t-t_o)}{\sqrt{\eta}} - \frac{c}{2F}\int_{-\infty}^{t}\left[\frac{E_{inc}(t')}{\sqrt{\eta}} - \frac{E_{inc}(t'-t_o)}{\sqrt{\eta}}\right]dt'\right\} \\
\frac{\tilde{V}_{rec,2}}{\sqrt{Z_c}} &= \frac{D}{2\sqrt{f_{g2}}}\left\{\frac{\tilde{E}_{inc}e^{-st_o}}{\sqrt{\eta}} - \frac{c}{2F}\frac{1}{s}\frac{\tilde{E}_{inc}}{\sqrt{\eta}}\left[1-e^{-st_o}\right]\right\}
\end{aligned}
\quad . \qquad \text{(A-3)}
$$

The equations become simpler if we use the wave parameters defined in [4],

$$
\begin{aligned}
\tilde{a} &= \frac{\tilde{V}_{inc}}{\sqrt{Z_2}} &&= \text{ incident power wave} \\
\tilde{b} &= \frac{\tilde{V}_{rec}}{\sqrt{Z_2}} &&= \text{ received power wave} \\
\tilde{\zeta} &= \frac{\tilde{E}_{inc}}{\sqrt{\eta}} &&= \text{ incident power flux density wave} \\
\tilde{\xi} &= \frac{r\,\tilde{E}_{rad}}{\sqrt{\eta}} e^{sr/c} &&= \text{ radiated radiation intensity wave}
\end{aligned}
\quad . \qquad \text{(A-4)}
$$

In that case, the transmission equations become

$$
\begin{aligned}
\xi(t) &= \frac{D}{4\pi\, c\sqrt{f_{g2}}}\left\{ a'(t-t_o) - \frac{c}{2F}\left[a(t) - a(t-t_o) \right] \right\} \\
\tilde{\xi} &= \frac{D}{4\pi\, c\sqrt{f_{g2}}}\left\{ s\,\tilde{a}\,e^{-st_o} - \frac{c}{2F}\tilde{a}\left[1 - e^{-st_o} \right] \right\}
\end{aligned}
\quad , \qquad \text{(A-5)}
$$

where $a'(t) = d/dt\left[a(t)\right]$. In reception, the equations are

$$
\begin{aligned}
b(t) &= \frac{D}{2\sqrt{f_{g2}}}\left\{ \zeta(t-t_o) - \frac{c}{2F}\int_{-\infty}^{t} \left[\zeta(t') - \zeta(t'-t_o) \right] dt' \right\} \\
\tilde{b} &= \frac{D}{2\sqrt{f_{g2}}}\left\{ \tilde{\zeta}\, e^{-st_o} - \frac{c}{2F}\frac{\tilde{\zeta}}{s}\left[1 - e^{-st_o} \right] \right\}
\end{aligned}
\quad . \qquad \text{(A-6)}
$$

This completes the simple model on boresight that includes the prepulse. An even more complete model is available in Mikheev et al [8], which includes near-field and off-boresight results.